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SOME COMPLETE-TYPE MAPS

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1. INTRODUCTION

As well known, in the topological category TOP uniform spaces are studied as the generalization of metric spaces, compact spaces and topological groups. In the fibrewise category TOP_B with the base space B , the study of fibrewise uniform space in TOP_B is found in James [5] Ch.3 and Konami-Miwa [6], [7]. Especially in [6] and [7], they studied the fibrewise uniform spaces by using coverings, and proved in [7] the equivalence of fibrewise uniform spaces by using entourages (in [5]) and their one (in [7]). The study of metrizable maps in TOP_B is found in [11], [9], [2], [8] and [3]. But for a metrizable map $p : X \rightarrow B$, the study of fibrewise uniformity on X has not been done.

In this paper, we announce the existence of fibrewise uniformities on some metrizable maps, and study the relations between the completeness induced by a trivial metric and the one defined by fibrewise uniformities. Further, we discuss the relations between completely metrizable maps and Čech-complete maps.

2. PRELIMINARIES

In this section, we refer to the notions and notations in Fibrewise Topology. For the definitions of undefined terms and notations, see [4], [3], [7] and [5].

Throughout this paper, we will use the abbreviation $nbds(s)$ for *neighborhood(s)*. Let B be a topological space with a fixed topology τ . For each $b \in B$, $N(b)$ is the family of all open nbds of b , and \mathbf{N} , \mathbf{Q} , \mathbf{R} and I are the sets of all natural numbers, all rational numbers, all real numbers and the unit interval, respectively. In this paper, we assume that (B, τ) is a regular space, all spaces are topological spaces and all maps are continuous.

For a map $p : X \rightarrow B$ and each $b \in B$, the *fibre* over b is the subset $X_b = p^{-1}(b)$ of X . Also for each subset B' of B , we denote $X_{B'} = p^{-1}B'$. For a filter \mathcal{F} on X , by a *b-filter* on X we mean a pair (b, \mathcal{F}) such that b is a limit point of the filter $p_*(\mathcal{F})$ on B , where $p_*(\mathcal{F})$ is the filter generated by the family $\{p(F) | F \in \mathcal{F}\}$. By an *adherence point* of a *b-filter* \mathcal{F} ($b \in B$) on X , we mean a point of the fibre X_b .

which is an adherence point of \mathcal{F} as a filter on X . For a projection $p : X \rightarrow B$ and $W \subset B$, we use the notation $X_W \times X_W = X_W^2$ and $X \times X = X^2$. For $D, E \subset X^2$, $D \circ E = \{(x, z) | \exists y \in X \text{ such that } (x, y) \in D, (y, z) \in E\}$ and $D(x) = \{y | (x, y) \in D\}$. For a family \mathcal{U} of subsets of a set X and a subset A of X , $\mathcal{U}|_A = \{U \cap A | U \in \mathcal{U}\}$.

Next, according to [11] let us refer to (completely) trivially metrizable maps. For a map $p : X \rightarrow B$ with a pseudometric ρ on X is called a *trivial metric* (*T-metric*, for short) on p if the restriction of ρ to every fibre $p^{-1}(b)$, $b \in B$, is a metric and $p^{-1}\tau \cup \tau_\rho$, where τ_ρ is the topology on X generated by ρ , is a subbase of the topology of X . A map $p : X \rightarrow B$ is called *trivially metrizable* (a *TM-map*, for short) if there exists a *T-metric* on p . A *T-metric* on a map $p : X \rightarrow B$ is called *complete* (a *CT-metric*, or short) if

- (*) For any b -filter \mathcal{F} , $b \in B$, on X containing elements of arbitrary small diameter, \mathcal{F} has adherence points.

A map $p : X \rightarrow B$ is called *completely trivially metrizable* (a *complete TM-map*, for short) if there exists a *CT-metric* on it.

A map $p : X \rightarrow B$ is called (resp. *closely*) *parallel* to a space Z if there exists an embedding $e : X \rightarrow B \times Z$ such that (resp. $e(X)$ is closed in $B \times Z$ and) $p = \pi \circ e$, where $\pi : B \times Z \rightarrow B$ is the projection (see [10]).

The following are proved in [11]: A map $p : X \rightarrow B$ is a *TM-map* if and only if p is parallel to a metrizable map, and p is a *complete TM-map* if and only if it is closely parallel to a completely metrizable (i.e., metrizable by complete metric) space.

Remark: By these, for a *TM-map* $p : X \rightarrow B$ there exists a metric space (M, ρ) and an embedding $e : X \rightarrow B \times M$ such that $p = \pi \circ e$. Then it is easy to see that we can define a *T-metric* (pseudometric) ρ' on X by $\rho'(x, y) = \rho(\pi \circ e(x), \pi \circ e(y))$, and vice versa. So, we can identify ρ on M and ρ' on X in the above meaning. In latter sections, we use the same notation ρ on M and on X .

We shall conclude this section by referring to fibrewise uniformities according to [7]. First, we recall the following definition.

Definition 2.1. Let $p : X \rightarrow B$ be a projection, and Δ be the diagonal of $X \times X$. A *fibrewise entourage uniformity* on X is a filter Ω on $X \times X$ satisfying the following four conditions:

- (J1) $\Delta \subset D$ for every $D \in \Omega$.
- (J2) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that $E \cap X_W^2 \subset D^{-1}$.
- (J3) Let $D \in \Omega$. Then for each $b \in B$ there exist $W \in N(b)$ and $E \in \Omega$ such that

$$(E \cap X_W^2) \circ (E \cap X_W^2) \subset D$$

- (J4) If $E \subset X \times X$ satisfies that for each $b \in B$ there exist $W \in N(b)$ and $D \in \Omega$ such that $D \cap X_W^2 \subset E$, then $E \in \Omega$.

Note that in [5] Section 12, a filter Ω on $X \times X$ satisfying (J1), (J2) and (J3) is called a *fibrewise uniform structure* on X . So, the notion of a fibrewise entourage uniformity is slightly stronger than one of a fibrewise uniform structure.

For a projection $p : X \rightarrow B$ and $W \in \tau$, let μ_W be a non-empty family of coverings of X_W . We say that $\{\mu_W\}_{W \in \tau}$ is a *system of coverings* of $\{X_W\}_{W \in \tau}$. (For this, we briefly use the notations $\{\mu_W\}$ and $\{X_W\}$). Let \mathcal{U} and \mathcal{V} be families of subsets of a set X . If \mathcal{V} refines \mathcal{U} in the usual sense, we denote $\mathcal{V} < \mathcal{U}$. Let us define the notion of fibrewise covering uniformity.

Definition 2.2. Let $p : X \rightarrow B$ be a projection, and $\mu = \{\mu_W\}$ be a system of coverings of $\{X_W\}$. We say that the system $\{\mu_W\}$ is a *fibrewise covering uniformity* (and a pair (X, μ) or $(X, \{\mu_W\})$ is a *fibrewise covering uniform space*) if the following conditions are satisfied:

- (C1) Let \mathcal{U} be a covering of X_W and for each $b \in W$ there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and $\mathcal{V} < \mathcal{U}$. Then $\mathcal{U} \in \mu_W$.
- (C2) For each $\mathcal{U}_i \in \mu_W, i = 1, 2$, there exists $\mathcal{U}_3 \in \mu_W$ such that $\mathcal{U}_3 < \mathcal{U}_i, i = 1, 2$.
- (C3) For each $\mathcal{U} \in \mu_W$ and $b \in W$, there exist $W' \in N(b)$ and $\mathcal{V} \in \mu_{W'}$ such that $W' \subset W$ and \mathcal{V} is a star refinement of \mathcal{U} .
- (C4) For $W' \subset W$, $\mu_{W'} \supset \mu_W|_{X_{W'}}$, where

$$\mu_W|_{X_{W'}} = \{\mathcal{U}|_{X_{W'}} | \mathcal{U} \in \mu_W\} \text{ and } \mathcal{U}|_{X_{W'}} = \{U \cap X_{W'} | U \in \mathcal{U}\}.$$

For a fibrewise entourage uniformity Ω on X , $D \in \Omega$ and $W \in \tau$, let $\mathcal{U}(D, W) = \{D(x) \cap X_W | x \in X_W\}$. Further let $\mu_W(\Omega)$ be the family of coverings \mathcal{U} of X_W satisfying that for each $b \in W$ there exist $W' \in N(b)$ and $D \in \Omega$ such that $W' \subset W$ and $\mathcal{U}(D, W') < \mathcal{U}$. Then the system $\mu(\Omega) = \{\mu_W(\Omega)\}$ is a fibrewise covering uniformity ([7] Proposition 3.7).

Conversely, for a fibrewise covering uniformity $\mu = \{\mu_W\}$, we can construct a fibrewise entourage uniformity $\Omega(\mu)$ as follows ([7] Construction 3.8): For $\mathcal{U} \in \mu_W$, $D(\mathcal{U}) = \bigcup \{U_\alpha \times U_\alpha | U_\alpha \in \mathcal{U}\}$. Let $\Omega(\mu)$ be the family of all subsets $D \subset X \times X$ satisfying the following condition:

$\Delta \subset D$, and for every $b \in B$ there exist $W \in N(b)$ and $\mathcal{U} \in \mu_W$ such that $D(\mathcal{U}) \subset D$.

Then $\Omega(\mu)$ is a fibrewise entourage uniformity ([7] Proposition 3.10). Further, we proved the following:

Theorem 2.3. ([7] Theorem 3.11) For a projection $p : X \rightarrow B$ and a fibrewise entourage uniformity Ω on X , we have $\Omega = \Omega(\mu(\Omega))$.

For a fibrewise entourage uniformity Ω on X and a fibrewise covering uniformity μ on X , let $\tau(\Omega)$ be the fibrewise topology induced by Ω ([5] Section 13) and $\tau(\mu)$ be the fibrewise topology induced by μ ([7] Proposition 3.8). Then $\tau(\Omega) = \tau(\mu(\Omega))$ and $\tau(\mu) = \tau(\Omega(\mu))$ ([7] Proposition 3.12).

3. FIBREWISE COVERING UNIFORMITIES ON TM -MAPS

For a TM -map $p : X \rightarrow B$ parallel to a metric space (M, ρ) , let $e : X \rightarrow B \times M$ be the embedding. For each $n \in \mathbb{N}$, let \mathcal{U}_n be the family $\{U(x, \frac{1}{n}) | x \in M\}$, where $U(x, \frac{1}{n}) = \{y \in M | \rho(x, y) < \frac{1}{n}\}$ and $\mathcal{W}_n = \{e^{-1}(B \times U) | U \in \mathcal{U}_n\}$. Then for each $W \in \tau$, let $\mu_W = \{\mathcal{U} | \bigcup \mathcal{U} = X_W \text{ and for each } b \in W \text{ there exists } n \in \mathbb{N} \text{ and } W' \in N(b) \text{ with } W' \subset W \text{ such that } \mathcal{W}_n|_{X_{W'}} < \mathcal{U}\}$.

Since μ_W and μ constructed above are induced by the metric ρ on M (on X), we call this $\mu = \{\mu_W\}$ a *fibrewise covering uniformity on X induced by the metric ρ* , and denoted by $\mu_\rho = \{\mu_W\}_\rho$. Further, by the construction of $\{\mathcal{W}_n | n \in \mathbb{N}\}$ in the above, we say that the family $\{\mathcal{W}_n | n \in \mathbb{N}\}$ is the *standard developable covering* (*sd-covering*, for short) on X induced by ρ . (Note that we exclusively use the notation $\{\mathcal{W}_n | n \in \mathbb{N}\}$ as *sd-covering* induced by ρ in this paper.)

Theorem 3.1. For a TM -map $p : X \rightarrow B$ with a T -metric ρ , the system $\mu_\rho = \{\mu_W\}_\rho$ is a fibrewise covering uniformity on X induced by ρ .

4. EQUIVALENCE OF SOME COMPLETENESS ON TM -MAPS

Definition 4.1. ([5] Definition 14.1) For a map $p : X \rightarrow B$, let Ω be a fibrewise entourage uniformity on X .

- (1) A subset M of X is said to be *D-small*, where $D \subset X^2$, if M^2 is contained in D .
- (2) A b -filer \mathcal{F} , where $b \in B$, is *Cauchy* if \mathcal{F} contains a D -small members for each $D \in \Omega$. (We call \mathcal{F} *J-Cauchy* with respect to Ω (w.r.t. Ω , for short), for convenience' sake.)

We shall define a new notion of Cauchy b -filter in fibrewise covering uniformity $\mu = \{\mu_W\}$ on X .

Definition 4.2. For a map $p : X \rightarrow B$, let $\mu = \{\mu_W\}$ be a fibrewise covering uniformity on X . A b -filer \mathcal{F} , where $b \in B$, is *Cauchy* if for each $W \in N(b)$ and $\mathcal{U} \in \mu_W$ there exist $F \in \mathcal{F}$ and $U \in \mathcal{U}$ such that $F \subset U$. (We call \mathcal{F} *CU-Cauchy* with respect to μ (w.r.t. μ , for short), for convenience' sake.)

Theorem 4.3. For a map $p : X \rightarrow B$, let Ω be a fibrewise entourage uniformity on X . Then for each $b \in B$, a b -filer \mathcal{F} is *J-Cauchy* w.r.t. Ω if and only if it is *CU-Cauchy* w.r.t. $\mu(\Omega)$.

For a space X , let $\Upsilon = \{\Phi_\alpha | \alpha \in \Lambda\}$ be a family of families of subsets of X . We say that a family Ψ of subsets of X is *subordinated* to the family Υ if for each $\alpha \in \Lambda$ there exists $U_\alpha \in \Phi_\alpha$ and $V \in \Psi$ such that $V \subset U_\alpha$.

Definition 4.4. Let $p : X \rightarrow B$ be a TM -map with a T -metric ρ .

(1)([11]) The map p is *complete* if for any b -filter \mathcal{F} , $b \in B$, on X subordinated to the sd -covering $\{\mathcal{W}_n | n \in \mathbb{N}\}$ induced by ρ , it has adherence points. (We call this “complete” *P-complete*, and also call this b -filter satisfying this condition *P-Cauchy* w.r.t ρ .)

(2)([5] Definition 14.10) The map p is *complete* if for each $b \in B$ any J -Cauchy b -filter \mathcal{F} w.r.t. $\Omega(\mu_\rho)$ converges. (We call this “complete” *J-complete*.)

Theorem 4.5. For a TM -map $p : X \rightarrow B$ with a T -metric ρ and each $b \in B$, a b -filter \mathcal{F} is a *P-Cauchy* w.r.t. ρ if and only if it is a *J-Cauchy* w.r.t. Ω_ρ .

5. COMPLETE TM -MAPS AND ČECH-COMPLETE MAPS

Definition 5.1. A T_2 -compactifiable map $p : X \rightarrow B$ is *Čech-complete* if for each $b \in B$, there exists a countable family $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ of open (in X) covers of X_b with the property that every b -filter \mathcal{F} which is subordinated to the family $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ has an adherence point.

Proposition 5.2. (1) ([1] Theorem 6.1) Every locally compact map is Čech-complete

(2) ([1] Theorem 4.1) For T_2 -compactifiable maps $p : X \rightarrow B$, $q : Y \rightarrow B$ and a perfect morphism $f : p \rightarrow q$, p is Čech-complete if and only if q is Čech-complete .

Lemma 5.3. Every TM -map $p : X \rightarrow B$ is a $T_{3\frac{1}{2}}$ -map.

By this lemma, every TM -map is $T_{3\frac{1}{2}}$ -compactifiable. For complete TM -maps, we can prove the following.

Theorem 5.4. If $p : X \rightarrow B$ is a complete TM -map, then p is Čech-complete.

6. MT -MAPS AND SOME PROBLEMS

About the relations of TM -maps and MT -maps, we have the following.

- (a) A closed TM -map is an MT -map.
- (b) There exists a compact MT -map which is not a TM -map.
- (c) There exists (complete) TM -maps which are not closed, so not MT -maps.

Theorem 6.1. If $p : X \rightarrow B$ is a closed TM -map, then p is an MT -map.

As discussed in section 5, there seems to exist many problems about relations between metrizable maps and completeness. As an attempt to the problems, we define a new notion of D -complete MT -maps. For an MT -map $p : X \rightarrow B$, we use the following notation: $\{\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}} | b \in B\}$ is a p -development, where $\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}}$ is a b -development. First, we recall some definitions and theorems of MT -maps according to [3].

Definition 6.2. (1)([3] Def. 2.8) For a map $p : X \rightarrow B$, a sequence $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ of open (in X) covers of X_b , $b \in B$, is said to be a b -development if for every $x \in X_b$ and every $U \in N(x)$, there exists $n \in \mathbb{N}$ and $W \in N(b)$ such that $x \in st(x, \mathcal{U}_n) \cap X_W \subset U$. The map p is said to have a p -development if it has a b -development for every $b \in B$.
(2)([3] Def. 2.9) A closed map $p : X \rightarrow B$ is said to be an MT -map if it is collectionwise normal and has a p -development.

Definition 6.3. For an MT -map $p : X \rightarrow B$ equipped with p -development $\{\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}} | b \in B\}$, we call p D -complete with respect to the p -development if for each $b \in B$ every b -filter \mathcal{F} subordinated to $\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}}$ has adherence points.

Problem 6.4. For an MT -map $p : X \rightarrow B$, let $\{\{\mathcal{U}_n(b)\}_{n \in \mathbb{N}} | b \in B\}$ be a p -development.

- (1) Is there a fibrewise (covering) uniformity on X related to the p -development?
- (2) If Problem (1) had an affirmative answer, then is the J -completion of p w.r.t. the fibrewise (covering) uniformity on X equivalent to D -completion?

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